

# The Eeckhout Condition and the Subgame Perfect Implementation of Stable Matching\*

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## Abstract

We investigate an extensive form sequential matching game of perfect information. We show that the subgame perfect equilibrium of the sequential matching game leads to the unique stable matching when the Eeckhout Condition (2000) for existence of a unique stable matching holds, regardless of the sequence of agents. This result does not extend to preferences that violate the Eeckhout Condition, even if there is a unique stable matching.

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# 1 Introduction

Gale and Shapley (1962) introduced a marriage problem which is described by a group of men and women, and their preferences over their potential mates.<sup>1</sup> They proved the existence of the stable matching by providing an algorithm which leads to a stable matching. We consider the issue of the implementation of stable matchings in relation to the following extensive form sequential matching game.<sup>2</sup> Men and women move sequentially according to a certain order. A typical agent can choose one of three possible actions at his or her move. The agent can either accept one among those who proposed to this agent at previous stages, or propose to a possible mate who moves at a later stage, or choose to be single. An agent has perfect information about the game structure, the others' preferences, and the actions of the agents who move in the previous stages. In addition, we concentrate on the marriage problems in which the preferences are strict so that each corresponding sequential matching game has a unique subgame perfect equilibrium(SPE). One may wonder whether the SPE outcome of a sequential matching game is a stable matching of the corresponding marriage problem. Eeckhout (2000) provided a sufficient condition which identifies a set of preferences guaranteeing a unique stable matching. Within such a domain of preference profiles, we show that the SPE of every sequential matching game, regardless of the order of moves, coincides with the unique stable matching. This result, however, does not extend to preference profiles that violate the Eeckhout Condition, even if there is a unique stable matching. We provide a counter example where there is a unique stable matching but the SPE outcome is not the stable matching.

## 2 An Extensive Form Matching Game

We first introduce some definitions and notations that are necessary to describe a marriage problem. For a description of an extensive form game, we refer readers to textbooks, for

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<sup>1</sup>For an introduction of the marriage problem, refer to Roth and Sotomayor (1990).

<sup>2</sup>The issue of the implementation of stable matchings has been investigated in Nash (undominated or strong Nash) equilibrium. Refer to Alcalde (1996), Kara and Sönmez (1996), Ma (1995), Shin and Suh (1996), and Sönmez (1997).

example, Osborne and Rubinstein (1994).

There are two non-empty and finite sets of agents,  $M = \{m_1, \dots, m_n\}$  for “men” and  $W = \{w_1, \dots, w_n\}$  for “women,” where  $M \cap W = \emptyset$  and  $\|M\| = \|W\| = n$ . Each agent  $i \in M \cup W$  has a complete, irreflexive, and transitive strict preference relation  $P_i$  defined over  $W \cup \emptyset$  for  $i \in M$  and over  $M \cup \emptyset$  for  $i \in W$ . We read  $aP_ib$  as “ $a$  is strictly preferred to  $b$  by agent  $i$ .” A *matching* is a function  $\mu : M \cup W \rightarrow M \cup W$  such that

$\mu(m) \in W \cup \emptyset$  for all  $m \in M$ ,  $\mu(w) \in M \cup \emptyset$  for all  $w \in W$ , and

$\mu(m) = w$  if and only if  $\mu(w) = m$  for all  $m \in M$  and  $w \in W$ .

Given a preference profile  $P = (P_i)_{i \in M \cup W}$ , a matching  $\mu$  is *individually rational* if for all  $i \in M \cup W$ ,  $\mu(i)P_i\emptyset$ . Given a matching  $\mu$  and  $P$ , a pair  $(m, w) \in M \times W$  *blocks* the matching  $\mu$  if  $wP_m\mu(m)$  and  $mP_w\mu(w)$ . A matching is *stable* if it is individually rational and does not allow any blocking pair.

We now introduce an extensive form sequential matching game. Let all agents be ordered as  $1, 2, \dots, \|M \cup W\|$  according to the order of their moves. Different orders of moves lead to different games, hence possibly different SPE outcomes.

**Stage 1.** Agent 1 may either propose to a possible mate or choose to be single.

**Stage 2.** Agent 2 observes agent 1’s move. If agent 1 proposed to agent 2, then agent 2 may either accept agent 1’s proposal, or (reject agent 1’s proposal and) propose to a potential mate in  $M \cup W \setminus \{1\}$ , or choose to be single. If agent 1 did not propose to agent 2, then agent 2 may only either propose to a potential mate in  $M \cup W \setminus \{1\}$ , or choose to be single.

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**Stage  $i$ .** After observing agents’ moves in all previous stages, agent  $i$  may either accept only one of the proposals directed to him/her (and reject the others if there are any), or (reject all proposals to him/her and) propose to a potential mate in  $M \cup W \setminus \{1, \dots, i\}$ , or choose to be single. Note that if agent  $i$  has no one to accept and no one to propose to, then

being single is the only choice left.

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**Stage  $\parallel M \cup W \parallel$ .** After observing all the agents' moves, agent  $\parallel M \cup W \parallel$  may either accept one of the proposals directed to him/her (and reject the others if there are any), or (reject all the previous proposals to him/her and) choose to be single.

All the agents who either choose to be single or are rejected will have no mates, and all the other agents will be matched to agents they proposed to or were accepted by. It is easy to see that such a sequential matching game is a well defined finite game with perfect information, and hence has a unique SPE under strict preferences.

### 3 The Main Result

The following condition on agents preferences, introduced by Eeckhout (2000), is sufficient for the existence of a unique stable matching:<sup>3</sup>

**The Eeckhout Condition:** *It is possible to rename (rearrange) agents so that*

- i) *For any agent  $m_i \in M$ ,  $w_i P_{m_i} w_j$  for all  $j > i$ ;*
- ii) *For any agent  $w_i \in W$ ,  $m_i P_{w_i} m_j$  for all  $j > i$ .*

Note that agent  $m_i$  may or may not prefer  $w_i$  to  $w_k$  for  $k < i$ . Under the Eeckhout Condition,  $\mu$  is the unique stable matching if and only if  $w_i = \mu(m_i)$  for all  $i$ . Our main result is:

**Theorem 1** *Suppose that agents' preferences satisfy the Eeckhout Condition. Then regardless of the order of agents' moves, the subgame perfect equilibrium of the sequential matching game leads to the unique stable matching.*

**Proof** We will prove the theorem by using an inductive argument. First, since  $m_1$  prefers  $w_1$  to any other woman under the Eeckhout Condition, it is a strictly dominant strategy for

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<sup>3</sup>See Clark (2005) for another sufficient condition, called the No Crossing Condition, which is interesting but stronger than the condition by Eeckhout (2000).

$m_1$  to propose to  $w_1$  if he moves before  $w_1$  or to accept  $w_1$  if  $w_1$  proposes to him. Similarly, it is a strictly dominant strategy for  $w_1$  to propose to  $m_1$  if she moves before  $m_1$  or to accept  $m_1$  if  $m_1$  proposes to her. Their strategies off the equilibrium path, such as  $w_1$ 's strategy after  $m_1$  did not propose to  $w_1$ , are irrelevant to the equilibrium outcome. Consequently,  $m_1$  and  $w_1$  will be matched in the SPE, regardless of the order of their moves in the sequential matching game.

For any  $k \leq n$ , assume that, for all  $i < k$ , (1) no  $m \in \{m_1, \dots, m_{i-1}\}$  proposes to  $w_i$  and no  $w \in \{w_1, \dots, w_{i-1}\}$  proposes to  $m_i$ , (2)  $m_i$  either proposes to  $w_i$  if he moves before  $w_i$  or accepts  $w_i$  if  $w_i$  proposes to  $m_i$ , and (3)  $w_i$  either proposes to  $m_i$  if  $w_i$  moves before  $m_i$  or accepts  $m_i$  if  $m_i$  proposes to  $w_i$ . Accordingly,  $m_i$  and  $w_i$  will be matched for all  $i < k$ , regardless of the order of their moves.

It is impossible for  $m_k$  to be matched with any  $w \in \{w_1, \dots, w_{k-1}\}$  and  $w_k$  to be matched with any  $m \in \{m_1, \dots, m_{k-1}\}$ . Under the Eeckhout Condition,  $m_k$  prefers  $w_k$  to any  $w \in \{w_{k+1}, \dots, w_n\}$ . Therefore,  $m_k$  will either propose to  $w_k$  if no  $m \in \{m_1, \dots, m_{k-1}\}$  proposes to  $w_k$  or accept  $w_k$  if  $w_k$  proposes to him. Similarly,  $w_k$  will either propose to  $m_k$  if no  $w \in \{w_1, \dots, w_{k-1}\}$  proposes to  $m_k$  or accept  $m_k$  if  $m_k$  proposes to her. Accordingly,  $m_k$  and  $w_k$  will be matched in the equilibrium, regardless their places to move.

By induction, the resulting SPE outcome coincides with the unique stable matching with  $m_i$  and  $w_i$  matched for all  $i \in \{1, \dots, n\}$ . **Q.E.D.**

## 4 A Counter Example

Can Theorem 1 be extended to preference profiles that violate the Eeckhout Condition? We provide a counter example with three men and three women to demonstrate that the answer to this question is negative, even if there is a unique stable match. We show that the SPE outcome of a sequential matching game in which all the men move before all the women is not the stable matching, but the SPE outcome of a sequential matching game in which all

the women move before all the men is the stable matching.<sup>4</sup>

There are three men and three women,  $M = \{m_1, m_2, m_3\}$  and  $W = \{w_1, w_2, w_3\}$ , with the following preferences:

$P_{m_1}$	$P_{m_2}$	$P_{m_3}$	$P_{w_1}$	$P_{w_2}$	$P_{w_3}$
$w_2$	$w_2$	$w_1$	$m_2$	$m_3$	$m_1$
$w_3$	$w_1$	$w_2$	$m_3$	$m_1$	$m_2$
$w_1$	$w_3$	$w_3$	$m_1$	$m_2$	$m_3$

which violate the Eeckhout Condition. Nevertheless, this marriage problem has a unique stable matching:

$$\mu^* = [(m_1, w_3), (m_2, w_1), (m_3, w_2)].$$

Consider the following order of moves:  $(m_1, m_2, m_3, w_1, w_2, w_3)$ , i.e.,  $m_1$  moves first,  $m_2$  moves next, etc. Let  $a_i$  be the action taken by agent  $i$ . The following strategy  $\sigma = (\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3}, \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3})$  is the unique SPE:

$$\begin{aligned} \sigma_{m_1} &= w_3; \\ \sigma_{m_2} &= \begin{cases} w_2 & \text{if } a_{m_1} \neq w_2 \\ w_1 & \text{if } a_{m_1} = w_2; \end{cases} \\ \sigma_{m_3} &= \begin{cases} w_1 & \text{if } a_{m_2} = w_2 \\ w_2 & \text{if } a_{m_2} \neq w_2; \end{cases} \\ \sigma_{w_1} &= \begin{cases} m_2 & \text{if } a_{m_2} = w_1 \\ m_3 & \text{if } a_{m_2} \neq w_1, \text{ and } a_{m_3} = w_1 \\ m_1 & \text{if } a_{m_2} \neq w_1, a_{m_3} \neq w_1 \text{ and } a_{m_1} = w_1 \\ w_1 & \text{otherwise;} \end{cases} \\ \sigma_{w_2} &= \begin{cases} m_3 & \text{if } a_{m_3} = w_2 \\ m_1 & \text{if } a_{m_3} \neq w_2, \text{ and } a_{m_1} = w_2 \\ m_2 & \text{if } a_{m_3} \neq w_2, \text{ and } a_{m_1} \neq w_2, \text{ and } a_{m_2} = w_2 \\ w_2 & \text{otherwise;} \end{cases} \\ \sigma_{w_3} &= \begin{cases} m_1 & \text{if } a_{m_1} = w_3 \\ m_2 & \text{if } a_{m_1} \neq w_3, \text{ and } a_{m_2} = w_3 \\ m_3 & \text{if } a_{m_1} \neq w_3, \text{ and } a_{m_2} \neq w_3, \text{ and } a_{m_3} = w_3 \\ w_3 & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to see that the strategies  $\sigma_{w_1}$ ,  $\sigma_{w_2}$ , and  $\sigma_{w_3}$  are sequentially rational. Since all the men move prior to all the women, no woman has any chance to propose. Hence the

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<sup>4</sup>Eeckhout (2000) claims that the Eeckhout Condition is also necessary for the uniqueness of a stable matching when there are three men and three women. Although our example demonstrates that this is not the case, this discrepancy does not change our results

choices available to a woman are: accepting a proposal, if there is any, or rejecting all other proposals to her. Therefore, for a woman, her best response is to accept a proposal from the man whom she prefers most among those who proposed to her. The strategies  $\sigma_{w_1}$ ,  $\sigma_{w_2}$  and  $\sigma_{w_3}$  describe such choices.

Now consider  $\sigma_{m_3}$ . Given  $w_1$ 's preferences, if  $m_2$  did not propose to  $w_1$ , then  $m_3$ 's proposal to  $w_1$  would be accepted by  $w_1$ . But if  $m_2$  proposes to  $w_1$ ,  $m_3$  has no chance of being matched to  $w_1$ . Hence, his best strategy is: propose to  $w_1$  if  $m_2$  did not propose to her and propose to  $w_2$  if  $m_2$  proposed to  $w_1$ .

Consider  $\sigma_{m_2}$ . If  $m_1$  did not propose to  $w_2$ , then  $m_2$ 's proposal to  $w_2$  would be accepted by  $w_2$ . Suppose  $m_1$  proposed to  $w_2$ . Since  $w_2$  prefers  $m_1$  to  $m_2$ ,  $m_2$ 's proposal to  $w_2$  would be rejected by  $w_2$ . Hence, his best strategy is: propose to  $w_2$  if  $m_1$  did not propose to  $w_2$  and propose to  $w_1$  if  $m_1$  proposed to  $w_2$ .

Consider  $\sigma_{m_1}$ . Two possible best strategies are proposing to  $w_2$  and  $w_3$ . Suppose he proposed to  $w_2$ . Then  $m_2$  will propose to  $w_1$  and  $m_3$  will propose to  $w_2$ . The resulting matching is  $[(m_1, \phi), (m_2, w_1), (m_3, w_2), (\phi, w_3)]$ . Suppose he proposed to  $w_3$ . Then  $m_2$  will propose to  $w_2$  and  $m_3$  will propose to  $w_1$ . The resulting matching is  $[(m_1, w_3), (m_2, w_2), (m_3, w_1)]$ . Therefore, proposing to  $w_3$  is his best strategy. Using the backward induction argument, we can show that  $\sigma$  is the SPE and the resulting matching is

$$\mu_\sigma = [(m_1, w_3), (m_2, w_2), (m_3, w_1)].$$

$\mu_\sigma$  is not a stable matching since it is blocked by  $(m_1, w_2)$ .

Now consider the following order of moves:  $(w_1, w_2, w_3, m_1, m_2, m_3)$ . For this order of moves where all the women move prior to all the men, one can show that the SPE outcome of this sequential matching game is the unique stable matching  $\mu^*$ .

This example demonstrates that different orders of moves lead to different SPE outcomes. Under a list of preference profiles which does not satisfy the Eeckhout Condition, a SPE outcome may not even be a stable matching. This raises an open question whether or not the Eeckhout Condition is also necessary for the SPE outcome of every sequential matching

game to be a stable matching of the corresponding matching problem. However, this does not exclude the possibility that, without the Eeckhout Condition, a stable matching can be a SPE outcome of a sequential game of a certain specific order of moves.



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